Finite temperature results on the 2D Ising model with mixed perturbation

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Abstract
A numerical study of finite temperature features of thermodynamical observables is performed for the lattice 2D Ising model. Our results support the conjecture that the finite size scaling analysis employed in the study of integrable perturbation of conformal field theory is still valid in the present case, where a non-integrable perturbation is considered.

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1. Introduction

Quantum field theories (QFT) in two spacetime dimensions at finite temperature are an appealing and complex subject which has attracted much attention from both the theoretical and the experimental point of view. It could be useful to recall that a finite temperature QFT can be obtained by considering the model on a cylinder with the radius of compactification proportional to the inverse of the physical temperature.

Since the 2D Ising model fuses together the simplicity and the full features of more complex models, we considered the most general, relevant perturbation of such model (both thermal and magnetic) in order to cope with the problem of finite temperature corrections to thermodynamic observables (in particular, as will be clear later, such corrections are nothing but the finite size scaling behaviour of the model as a function of the radius of compactification).

In this respect we recall that, taking advantage of integrability, the finite temperature study of the pure magnetic perturbation of the model was performed in [1] where an expression for the one-point functions was proposed in the framework of form factors. Such results were found to be in agreement with the high precision numerical analysis of [2].

In this paper, we investigate the finite temperature features of the model in the region of the coupling constants (h representing the coupling to the spin operator, and $t = |\beta - \beta_c|/\beta$ the

deviation from the critical temperature) where the thermal perturbation is small with respect to the magnetic one, $t/h^{8/15} \to 0$.

The main idea underlying our approach is to evaluate the finite size scaling (FSS) on the cylindric geometry of the main thermodynamic observables (magnetization, free energy, magnetic susceptibility) by means of high precision numerical data coming from the numerical diagonalization of the transfer matrix. Consequently, the finite temperature corrections to such observables, i.e. the FSS behaviour, are given by the standard thermodynamic Bethe ansatz (TBA) prescriptions.

Such an approach is known to work when the perturbations are separately switched on, giving rise to integrable QFTs (see [1–3]). The contemporary presence of both of them destroys the integrability; however, since we are in the region of the phase diagram where $t/h^{8/15} \to 0$, we postulate that the functional forms coming from the TBA still hold.

We showed that this expectation is correct indeed (at least within the precision of our data), and as a side result we were able to evaluate the infinite volume quantities with higher precision than before [4]. It is important to point out that such functional form can be rigorously derived only when integrable theories are concerned; however, we expect that when integrability is lost they still hold under pair creation threshold which is just the regime explored in our simulations.

The work is organized as follows: in section 2 we briefly present the model and the numerical transfer matrix technique; section 3 is devoted to the study of the FSS of some observables of the Ising model in the magnetic field; in section 4 we present our numerical results; finally, in section 5 we give our conclusion.

2. The model and the technique

The Ising model in a magnetic field at an arbitrary temperature is defined by the partition function

$$Z(\beta, h_\ell) = \sum_{\sigma_i = \pm 1} e^{\beta \sum_{\langle n,m \rangle} \sigma_n \sigma_m + h_\ell \sum_n \sigma_n} \tag{1}$$

where the spin variable σ_n takes the values ± 1; the notation $\langle n, m \rangle$ represents the nearest-neighbour sites on the lattice; the sites are labelled by $n = (n_o, n_1)$ and the two sizes of the square lattice are L and R (they are taken to be different because our transfer matrix calculations will treat the two directions asymmetrically); the total number of sites of the lattice will be denoted as $N = LR$.

The coupling β is the inverse of the temperature, while the magnetic perturbation is introduced by the coupling $h_\ell \equiv H\beta$, where H is the magnetic coupling. This model undergoes a second-order phase transition when $h_\ell = 0$ and β reaches its critical value β_c [5]

$$\beta_c \equiv \tfrac{1}{2} \log(\sqrt{2} + 1) = 0.440\,6868\ldots. \tag{2}$$

The observables which we shall consider are the following:
the *magnetization*

$$M(\beta, h_\ell) = \frac{1}{N} \frac{\partial}{\partial h_\ell} (\log Z(\beta, h_\ell)) = \frac{1}{N} \left\langle \sum_i \sigma_i \right\rangle \tag{3}$$

the *magnetic susceptibility*

$$\chi(\beta, h_\ell) = \frac{\partial M(\beta, h_\ell)}{\partial h_\ell} \tag{4}$$

the *free energy*

$$f(\beta, h_\ell) = \frac{1}{N} \log Z(\beta, h_\ell) - f_b \tag{5}$$

where the bulk constant $f_b = 0.929\,695\,3982\ldots$ can be found in [5].

Since we will often refer to exact results coming from the QFT, which describes the Ising model in the scaling limit, we recall that its action is given by

$$\mathcal{A}_{\mathrm{IM}} = \mathcal{A}_{\mathrm{CFT}} + \tau \int \mathrm{d}^2 x\, \epsilon(x) + h \int \mathrm{d}^2 x\, \sigma(x) \tag{6}$$

where the first term is the action of the minimal unitary CFT describing the critical point of the model [6], h is the coupling constant associated with the magnetic field and τ gives the deviation from the critical temperature (for a review on this subject see [7]). Such a theory is integrable either when $h = 0$, $\tau \neq 0$ and one can prove that it describes a massive Majorana fermion, or in the case $h \neq 0$, $\tau = 0$ which turns out to be a nontrivial theory of eight self-conjugated particles whose spectrum follows the numerology of the exceptional group E_8 as proved by Zamolodchikov in [8]. Simple arguments from the CFTs are sufficient to conclude that in the general case $h \neq 0$, $\tau \neq 0$ integrability[3] is lost [9].

In order to analyse the finite temperature behaviour of the model our choice was to employ the numerical transfer matrix technique. The basic idea is to rewrite the Boltzmann weight by means of the so-called transfer matrix T, which turns out to be a positive symmetric $2^R \times 2^R$ matrix such that

$$Z(\beta, h_\ell) = \sum_{\sigma_i = \pm 1} \mathrm{e}^{\beta \sum_{\langle n,m \rangle} \sigma_n \sigma_m + h_\ell} = \operatorname{tr} T^L = \sum_i \lambda_i^L \tag{7}$$

where the λ_i are the (real) eigenvalues of T (details can be found in [10–12]). The numerical computation of eigenvalues and eigenvectors of the transfer matrix enables us to compute all the observables we need, provided that we specify the values of h_ℓ, β and R. As an example, the free energy

$$f(\beta, h_\ell) = \frac{1}{RL} \log Z(\beta, h_\ell) \tag{8}$$

can be written, in the limit $L \to \infty$, as

$$f(\beta, h_\ell) \sim \frac{1}{R} \log \lambda_{\max} \tag{9}$$

which is exactly what we need for the numerical study of the thermodynamic observable as function of the length of the cylinder R.

3. Finite temperature results, the integrable case

Taking advantage of the integrability of the model (6) when ($\tau = 0, h \neq 0$), Delfino [1] applied the form factor method to propose a compact expression for the finite size (i.e. the finite temperature) corrections of the one-point functions of the Ising model with pure magnetic perturbation (see also [13–15] for a discussion on the finite temperature analysis of 2D integrable quantum field theory).

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[3] We would like to stress that we will always refer to the integrability of the continuum theory because we will use exact results coming from the latter. We are not at all interested in the integrability of the lattice model since our approach is numeric and hence it is not related to the lattice integrability of (1).

Table 1. Numerical values of the constants.

$\mathcal{C}_1 = 4.404\,908\,58\ldots$
$\mathcal{C}_2 = 7.127\,291\,79\ldots$
$\mathcal{C}_3 = 8.761\,556\,05\ldots$
$A_f = 1.197\,733\,38\ldots$

Using these results, a given one-point function $\langle\Phi\rangle_R$ calculated on a cylinder of radius R is related to its infinite plane value $\langle\Phi\rangle_\infty$ by

$$\frac{\langle\Phi\rangle_R}{\langle\Phi\rangle_\infty} = 1 + \frac{1}{\pi}\sum_{i=1}^{3}\mathcal{A}_i^\Phi K_0(r_i) + \mathcal{O}(e^{-2r_1}) \tag{10}$$

where $r_i = m_i R$ and m_i, with $i = 1, 2, 3$, are the first three masses of Zamoldchikov's mass spectrum [8].

The main result of [1] is to compute exactly the universal constants A_i^Φ. A numerical check of (10) has been performed by Caselle *et al* [2] using the transfer matrix technique to study the Ising model perturbed with a magnetic field. As already pointed out in [1], the one-point function of the perturbing operator σ can be obtained by TBA calculations. In fact, following TBA prediction for the FSS behaviour of the free energy, it is possible to write

$$f(R) = f(\infty) + \sum_{i=1}^{3}\frac{E(r_i)}{R} + \mathcal{O}(e^{-2r_1}) \tag{11}$$

where $E(r_i)$ is given by

$$E(r_i) = -\frac{m_i}{\pi}K_1(r_i) \tag{12}$$

and $K_1(r)$ is the modified Bessel function of the second kind.

Hence, for the free energy we have

$$\frac{f(R)}{f(\infty)} = 1 - \sum_{i=1}^{3}\frac{\mathcal{A}_i^f}{\pi}\frac{K_1(r_i)}{r_i} + \mathcal{O}(e^{-2r_1}) \qquad \mathcal{A}_i^f = \frac{\mathcal{C}_i^2}{A_f} \tag{13}$$

which gives us the first few orders (under the lowest particle pair creation threshold) of the finite size corrections; the constants $\mathcal{C}_i$ are given by Zamolodchikov's mass spectrum (see [8]). A first derivative with respect to the magnetic field just gives us expression (10) for the magnetization[4]

$$\frac{\langle\sigma\rangle_R}{\langle\sigma\rangle_\infty} = 1 + \sum_{i=1}^{3}\frac{\mathcal{A}_i^\sigma}{\pi}K_0(r_i) + \mathcal{O}(e^{-2r_1}) \qquad \mathcal{A}_i^\sigma = \frac{\mathcal{C}_i^2}{2A_f} \tag{14}$$

then a further derivative gives us the functional form of the finite size corrections for the magnetic susceptibility

$$\frac{\chi(R)}{\chi(\infty)} = 1 - \sum_{i=1}^{3}\frac{\mathcal{A}_i^\chi}{\pi}[K_0(r_i) - 8r_i K_1(r_i)] + \mathcal{O}(e^{-2r_1}) \qquad \mathcal{A}_i^\chi = \frac{\mathcal{C}_i^2}{2A_f}. \tag{15}$$

The exact values of the constants are given in table 1.

The aim of the next section is twofold: on one hand, we perform a numerical check for the finite size formulae for free energy and magnetic susceptibility in the case of the integrable

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[4] The constants $\mathcal{A}_i^\sigma$ coincide with the coefficients A_i^σ found by Delfino with the form factor approach in [1].

model (for internal energy and magnetization this was proved in [2]); on the other hand, we check that all the functional forms also hold in the case of the mixed perturbation considered in [4]. The latter case implies that the constants acquire a nontrivial dependence with respect to both h_ℓ and t. Their functional form can be derived by the same scaling arguments employed in [4].

4. Numerical analysis

In order to check the behaviour of our data coming from the numerical diagonalization of the transfer matrix, we use the effective procedure presented in our previous work [4] and in [16].

First, for each value of h_ℓ we fit the value of $\langle \Phi \rangle_R$ according to the functional form of the finite size corrections found in the previous section. As an example we can consider the susceptibility, hence the fitting function is given by (see the previous section)

$$\frac{\chi(R)}{\chi(\infty)} = 1 - \sum_{i=1}^{3} \frac{\mathcal{A}_i^\chi}{\pi} [K_0(r_i) - 8r_i K_1(r_i)] + \mathcal{O}(e^{-2r_1}) \tag{16}$$

where we keep the quantities $\mathcal{A}_i^\chi$ and $\chi(\infty)$ as free parameters. Our fitting procedure follows a very stringent list of acceptance criteria as presented in [4]. The first nontrivial result is that the fits are acceptable (for all the observables considered here) even if the temperature is not set to its critical value. In this case the behaviour of the observables as a function of R cannot be proved to hold rigorously because the corresponding QFT is not integrable. Notwithstanding this, in the regime $t/h^{8/15} \to 0$ considered here and within our numerical precision, such functional forms continue to hold.

To the best of our knowledge this is the first evidence about the validity of formulae like that of the previous section for non-integrable theories. It could suggest that, when small perturbations of the integrable QFTs are considered, then the finite temperature behaviour of the thermodynamic observables remains the same and the only price to pay is a change in the constants $\mathcal{A}_i^\phi$ which acquire a nontrivial dependence with respect to the couplings t and h_ℓ. The following quantitative results were found:

(i) The numerical results for the infinite R observables $f(\infty)$, $\langle \sigma \rangle_\infty$, $\chi(\infty)$ are not only in complete agreement with respect to the estimates of [4], but they also show an improvement in their precision of about one order of magnitude. As an example of this we quote the following results for the susceptibility:

- $\beta = \beta_c, h = 0.07$

$$\chi(\beta_c, 0.07) = 0.757\,4405(3) \tag{17}$$

    compared to the value obtained in [4]

$$\chi(\beta_c, 0.07) = 0.757\,441(1) \tag{18}$$

- $\beta = 0.4349\ldots, h = 0.07$

$$\chi(0.07, 0.4349\ldots) = 0.846\,5333(3) \tag{19}$$

    compared to the value obtained in [4]

$$\chi(0.07, 0.4349\ldots) = 0.846\,533(1). \tag{20}$$

We stress that it is remarkable that such an improvement comes independently of the fact that β is fixed at its critical value or not. Moreover, such an improvement in precision could be expected since in this latter case we used the functional form of the finite temperature corrections instead of fitting with a set of generic exponentially decaying functions.

(ii) As anticipated, we supposed that the constants $\mathcal{A}_i^\phi$ are functions of t and h_ℓ. In order to test such a conjecture, we first define the following subtracted quantities in order to discard all the corrections which depend only on the magnetic field h_ℓ,

$$\mathcal{A}_i^\phi(t)|_{h_\ell} - \mathcal{A}_i^\phi(0)|_{h_\ell} = \mathcal{J}_{1,i}^\phi(h_\ell)t + \mathcal{J}_{2,i}^\phi(h_\ell)t^2 + \mathcal{J}_{3,i}^\phi(h_\ell)t^3 + \cdots. \qquad (21)$$

As pointed out in [4], it is a crucial step in order to compute the leading corrections due to the thermal perturbation. In this way we are able to fit our data with (21). The precision of our data enable us to compute only the linear term of such an expansion.

By means of the expansion for the scaling functions we are in the position to fit the amplitude $\mathcal{J}_1^{\mathcal{O}}(h_\ell)$ as a function of the magnetic field alone. We recall that, in principle, the functional form of the $\mathcal{J}_{1,i}^\phi(h_\ell)$ is nontrivial to be obtained; however, they can be determined by the same renormalization group and CFT arguments given in [2, 4] in order to construct the scaling functions of the thermodynamic observables. The results are

- *Free energy*: the fit with the scaling function

$$\mathcal{J}_{1,1}^f(h_\ell) = \frac{\mathcal{W}_1}{h_\ell^{\frac{8}{15}}} + \mathcal{W}_2 h_\ell^{\frac{2}{15}} + \mathcal{W}_3 h_\ell^{\frac{8}{15}} + \cdots \qquad (22)$$

and we can give a reliable estimate only of the leading term of the expansion

$$\mathcal{W}_1 = 90(8). \qquad (23)$$

- *Magnetization*: the fit with the scaling function

$$\mathcal{J}_{1,1}^\sigma(h_\ell) = \frac{\mathcal{R}_1}{h_\ell^{\frac{8}{15}}} + \mathcal{R}_2 h_\ell^{\frac{2}{15}} + \mathcal{R}_3 h_\ell^{\frac{8}{15}} + \cdots \qquad (24)$$

gives us

$$2.09 < \mathcal{R}_1 < 2.15. \qquad (25)$$

The other observables were outside our numerical precision.

(iii) Finally, a further numerical check is given by the analysis of the data at the critical temperature β_c. In this case the scaling function for the coefficients $\mathcal{A}_1^\phi$ depends only on the magnetic field h_ℓ and is given by

$$\hat{\mathcal{A}}_1^\Phi(h_\ell) = \mathcal{A}_1^\Phi\left(1 + b_{1,\Phi}|h_\ell|^{\frac{16}{15}} + b_{2,\Phi}|h_\ell|^{\frac{22}{15}}\right) + \cdots. \qquad (26)$$

Employing them as fitting functions, we find

$$8.07 < \mathcal{A}_1^\chi < 8.20 \qquad (27)$$

$$16.12 < \mathcal{A}_1^f < 16.28 \qquad (28)$$

where the numerical values of the amplitudes perfectly agree with their theoretical counterparts: $\mathcal{A}_1^\chi = 8.099\,97\ldots$ and $\mathcal{A}_1^f = 16.1999\ldots$ computed according to the formulae given in the previous section. Such results can be considered as further checks (besides the results of [2] where magnetization and internal energy were considered) of the predictions coming from the form factor approach.

The complete list of the numerical values of the parameters we used for the numerical diagonalization of the transfer matrix is given in table 2. A significant sample of the data employed in the computation of finite temperature corrections is given in table 3.

Table 2. Value of the parameters used for the diagonalization of the transfer matrix.

t	h_ℓ	R
0.013 0481	0.01	9
0.011 9726	0.02	10
0.010 8971	0.03	11
0.008 7367	0.04	12
0.007 6518	:	:
0.006 5668		
0.004 3875	0.17	17
0.002 1985	0.18	18
0.001 0992	0.19	19
0.0	0.20	20

Table 3. Magnetic susceptibility as a function of the compactification radius R ($h_\ell = 0.07$, $\beta = 0.4349\ldots$).

R	χ
9	0.860 965 432 368
10	0.852 426 578 479
11	0.848 936 912 437
12	0.847 512 060 489
13	0.846 931 329 088
14	0.846 695 116 084
15	0.846 599 363 455
16	0.846 560 454 648
17	0.846 544 860 407
18	0.846 538 199 957
19	0.846 535 213 111
20	0.846 532 714 557

5. Conclusions

Let us recall our main results. Our numerical study confirms the conjecture that the functional forms of the finite temperature corrections derived by means of the thermodynamic Bethe ansatz for the Ising model with magnetic perturbation at $t = 0$ ($T = T_c$) still hold when $t \neq 0$ supposing that $t/h^{8/15} \to 0$. We stress that the functional form of finite temperature corrections works well even if integrability is lost because we explored a region of the phase space where inelastic effects can be neglected. As explained in [17] the thermal perturbation induces just a correction to the mass spectrum of the integrable Ising field theory in magnetic field; no other effects due to the loss of integrability are expected.

Finally, it could be worth noting that the high precision of our data allowed us to achieve the following numerical results: first we obtained the infinite volume estimates of the observables with typically one order of magnitude of precision more than the method previously employed in [4]; second we get information about the first subleading correction due to the reduced temperature.

References

[1] Delfino G 2001 One-point functions in integrable quantum field theory at finite temperature *J. Phys. A: Math. Gen.* **34** L161 (*Preprint* hep-th/0101180)

[2] Caselle M and Hasenbusch M 2002 Finite temperature corrections in 2D integrable models *Nucl. Phys.* B **639** 549 (*Preprint* hep-th/0204088)

[3] Caselle M, Hasenbusch M, Pelissetto A and Vicari E 2001 The critical equation of state of the 2D Ising model *J. Phys. A: Math. Gen.* **34** 2923 (*Preprint* cond-mat/0011305)

[4] Grinza P and Rago A 2003 Study of the 2d Ising model with mixed perturbation *Nucl. Phys.* B **651** 387 (*Preprint* hep-th/0208016)

[5] Onsager L 1944 Crystal statistics: 1. A two-dimensional model with an order disorder transition *Phys. Rev.* **65** 117

[6] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 Infinite conformal symmetry in two-dimensional quantum field theory *Nucl. Phys.* B **241** 333

[7] Delfino G 2003 Integrable field theory and critical phenomena: the Ising model in a magnetic field *Preprint* hep-th/0312119

[8] Zamolodchikov A B 1989 Integrals of motion and S matrix of the (scaled) T = T(C) Ising model with magnetic field *Int. J. Mod. Phys.* A **4** 4235

[9] Mussardo G 1992 Off critical statistical models: factorized scattering theories and bootstrap program *Phys. Rep.* **218** 215

[10] Kramers H A and Wannier G H 1941 *Phys. Rev.* **60** 252

[11] Baxter R J 1982 *Exactly Solvable Models in Statistical Mechanics* (London: Academic)

[12] Nightingale M P 1990 *Finite Size Scaling and Numerical Simulation of Statistical Systems* ed V Privman (Singapore: World Scientific)

[13] Leclair A, Lesage F, Sachdev S and Saleur H 1996 Finite temperature correlations in the one-dimensional quantum Ising model *Nucl. Phys.* B **482** 579 (*Preprint* cond-mat/9606104)

[14] Mussardo G 2001 On the finite temperature formalism in integrable quantum field theories *J. Phys. A: Math. Gen.* **34** 7399 (*Preprint* hep-th/0103214)

[15] Leclair A and Mussardo G 1999 Finite temperature correlation functions in integrable QFT *Nucl. Phys.* B **552** 624 (*Preprint* hep-th/9902075)

[16] Caselle M and Hasenbusch M 2000 Critical amplitudes and mass spectrum of the 2D Ising model in a magnetic field *Nucl. Phys.* B **579** 667 (*Preprint* hep-th/9911216)

[17] Delfino G, Mussardo G and Simonetti P 1996 Non-integrable quantum field theories as perturbations of certain integrable models *Nucl. Phys.* B **473** 469 (*Preprint* hep-th/9603011)